L25 March 17 Connected

Monday, March 16, 2015 8:35 PM

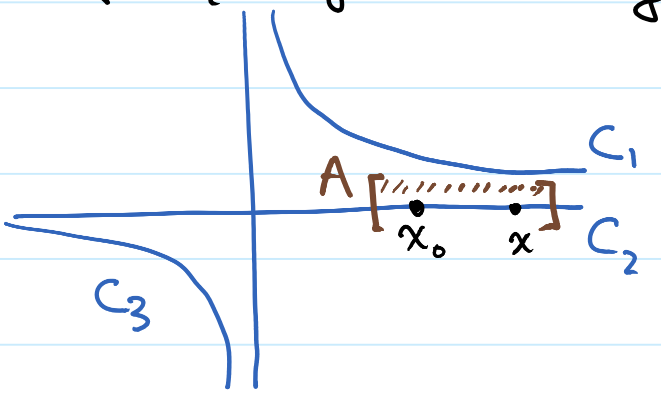
Connected component of $x_0 \in X$

① C is the maximal / largest connected subset of X containing x_0

② $C = \bigcup \{\text{connected subsets containing } x_0\}$

③ Define $\sim$ on X by $x \sim y$ if
$\exists$ connected $A \subset X$ such that $x, y \in A$
Then $C = [x_0]$

Example. $X = \{(x,y) \in \mathbb{R}^2 : xy = 0 \text{ or } xy = 1\}$



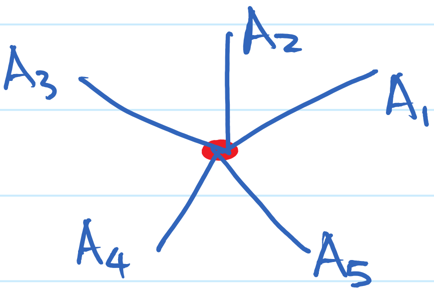
Intuitive picture

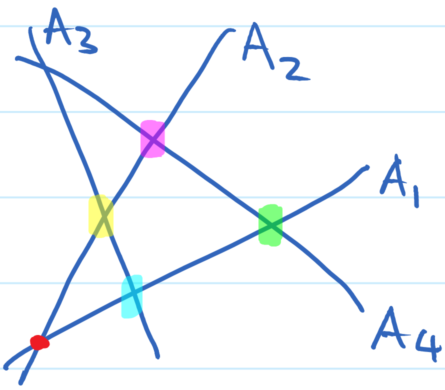
Qu. How do we know that C_1, C_2, C_3 are connected.

Qu. What must we do when we have 3 def's ?

Theorem Let $A_\alpha \subset X$ be connected subsets with
(i) $\bigcap_\alpha A_\alpha \neq \phi$ or (ii) $A_\alpha \cap A_\beta \neq \phi$ $\forall$ pair α, β

Then $\bigcup_\alpha A_\alpha$ is connected





① ⇒ ② By condition (i),

$$\bigcup\{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$$

is a connected set containing x_0.

② ⇒ ③ First, we need condition (ii)

to show that $x \sim y$ and $y \sim z \Rightarrow x \sim z$

Then, we show

$$[x_0] = \bigcup\{A \subset X : x_0 \in A \text{ and } A \text{ is connected}\}$$

"$\supset$" If $x \in$ RHS then $\exists\, A \subset X$ with

$x_0 \in A$ and A is connected at $x \in A$

By definition of $\sim$, $x \sim x_0$, $\therefore x \in [x_0]$

"$\subset$" Similar

③ ⇒ ① By def of $\sim$ and maximality of C

$x \in [x_0] \Rightarrow x \in A \subset C$

$\therefore [x_0] \subset C$

On the other hand, $x \in C \Rightarrow x \sim x_0$

$C \subset [x_0]$

Theorem If each X_α is connected

then $\prod X_\alpha$ is connected

Explore. Write $\prod X_\alpha = Cl(\bigcup_{\lambda \in I} A_\lambda)$ where

$\{A_\lambda\}$ satisfies (ii), use a later theorem

on the closure.

Note: (i) is a particular case of (ii),

Sufficient to show (ii) $\Rightarrow A = \bigcup_\alpha A_\alpha$ is connected

Let $S \subset A$ be both open and closed in A

$\therefore$ $S \cap A_\alpha$ is both open and closed $\forall \alpha$

$\therefore$ $\forall \alpha \in I$, $S \cap A_\alpha = \phi$ or $S \cap A_\alpha = A_\alpha$

$$\Downarrow ?$$

$$\underbrace{\forall \alpha \in I \quad S \cap A_\alpha = \phi} \quad \text{or} \quad \underbrace{\forall \alpha \in I \quad S \cap A_\alpha = A_\alpha}$$

$S = S \cap A = S \cap \left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} S \cap A_\alpha = \phi$ or $S = \cdots\cdots = \bigcup_{\alpha \in I} S \cap A_\alpha = \bigcup_{\alpha \in I} A_\alpha = A$

In ? above, logical, it may happen

$S \cap A_\alpha = \phi$ for some α; while $S \cap A_\alpha = A_\alpha$ for other α.

We need the assumption $A_\alpha \cap A_\beta \neq \phi$ $\forall$ pair α, β

Suppose $\exists$ particular α with $S \cap A_\alpha = \phi$

Then $S \cap (A_\alpha \cap A_\beta) \subset A_\alpha \cap A_\beta = \phi$

$$(S \cap A_\beta) \cap A_\alpha = \begin{cases} A_\beta \cap A_\alpha \neq \phi \\ \phi \cap A_\alpha = \phi \end{cases}$$

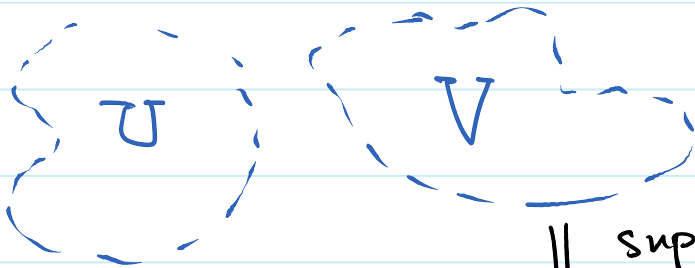
Thus $\forall$ arbitrary $\beta \in I$, $S \cap A_\beta = \phi$

Using the contrapositive, we get

$\exists \beta$ with $S \cap A_\beta = A_\beta \Longrightarrow \forall \alpha,\ S \cap A_\alpha = A_\alpha$

More about Connected components

Suppose X is disconnected

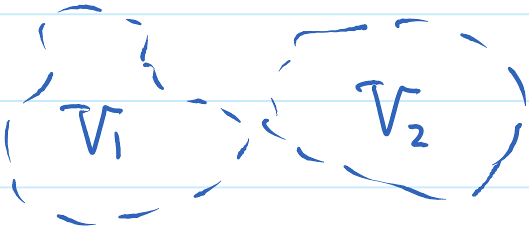


U, V both open & closed in X

connected?

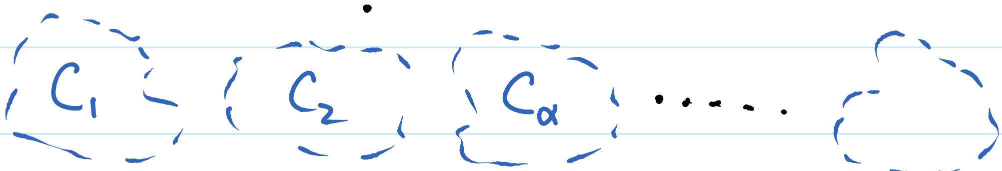
Suppose yes

suppose disconnected



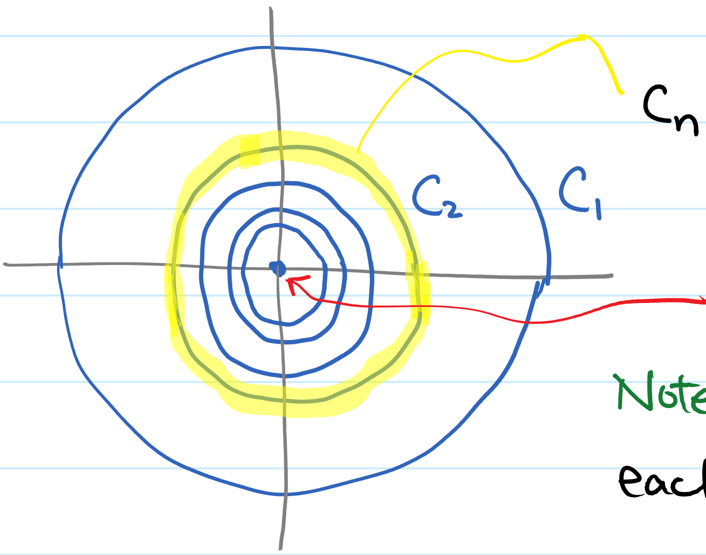
V_1, V_2 both open & closed in V and in X

⋮



Qu: Is each C_α both open and closed in X ?

Example. $X = \bigcup_{0<n\in\mathbb{Z}} \underbrace{\{(x,y)\in\mathbb{R}^2 : x^2+y^2=\tfrac{1}{n^2}\}}_{C_n} \cup \underbrace{\{(0,0)\}}_{C_\infty}$



$C_n = (\text{open}) \cap X, \quad 1 \le n \in \mathbb{Z}$

$\therefore$ open in X

C_∞ is not open

Note. In this example, each C_n is closed in X

Exercise. If $X = C_1 \cup \cdots \cup C_n$ has finitely many connected components then each C_k is both open and closed.

Theorem Let A be a connected subset in X and let $A \subset B \subset \bar{A}$. Then B is connected

As a result, each connected component C_α is closed.

Because $\bar{C_\alpha}$ is connected and $\bar{C_\alpha} \supset C_\alpha$.

By maximality of C_α, $\bar{C_\alpha} = C_\alpha$ and it is closed.

Proof.

Let $S \subset B$ be both open and closed in B

$\therefore\ S = G \cap B = F \cap B$ where $G, X \setminus F \in \mathcal{J}$

$S \cap A = G \cap A = F \cap A$ is both open closed in A

$\therefore\ \underset{\substack{\| \\ G \cap A}}{S \cap A} = \phi$ or $\underset{\substack{\| \\ F \cap A}}{S \cap A} = A$

$\therefore\ A \subset \underbrace{X \setminus G}_{\text{closed}}$ $\qquad \therefore\ \underset{\text{closed}}{F} \supset A$

$\therefore\ \underset{\substack{\cup \text{ given} \\ B}}{\bar{A}} \subset X \setminus G$ $\qquad \therefore\ F \supset \underset{\substack{\cup \text{ given} \\ B}}{\bar{A}}$

$\therefore\ S = G \cap B = \phi$ or $S = F \cap B = B$

Example. $O(n) = \{ n\times n \text{ orthogonal matrices} \} \subset \mathbb{R}^{n^2}$

$= \{ Q \in \mathbb{R}^{n^2} : Q^TQ = QQ^T = I \}$

Consider a function $f : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n(n+1)/2}$

$A \longmapsto \underbrace{A^TA}_{\text{symmetric}}$

continuous

Qu. What is $O(n)$ in terms of f ?

$O(n) = f^{-1}(I)$

The pre-image of a continuous function

No conclusion about its connectedness.

Consider $g : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$

$A \longmapsto \det(A)$

Clearly, g is continuous and

$g|_{O(n)} : O(n) \longrightarrow \{-1, 1\}$ is surjective

Thus, $O(n)$ is disconnected

Qu. What about $SO(n) = g^{-1}(1)$?

It is indeed path connected

A space X is **path connected** if

$\forall\ x_0, x_1 \in X\ \exists$ continuous $\gamma : [0,1] \longrightarrow X$

such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Theorem X is path connected $\Rightarrow$ X is connected

* The image $\gamma([0,1]) \subset X$ is connected
* $\forall\, x_0, x_1 \in X \quad x_0 \sim x_1$

Hence $X = [x_0]$, only one component

Explore. How to show $SO(n)$ is path connected?

Consider $U(n) = \{n \times n \text{ unitary matrices}\} \subset \mathbb{C}^{n^2}$

$$= \{A \in \mathbb{C}^{n^2} : A^* A = AA^* = I\}$$

Now, $g : \mathbb{C}^{n^2} \longrightarrow \mathbb{C} : M \longmapsto \det(M)$

is still continuous but the surjection is

$$g|_{U(n)} : U(n) \longrightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

Cannot conclude that $U(n)$ is disconnected.

Exploration.

① Show that $U(1)$ is the circle

② Show that $U(2)$ is homeomorphic to

quaternion $= \{a + bi + cj + dk : i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j\}$

③ Find out why $U(n)$ is connected

Locally Connected

Qu. Do you still remember how to define a local topological property?

A space X is locally connected if at every $x \in X$, $\exists$ local base of connected nbhds.

That is, $\forall\ x \in X\ \ \exists\ \mathcal{U}_x \subset \mathcal{J}$ such that

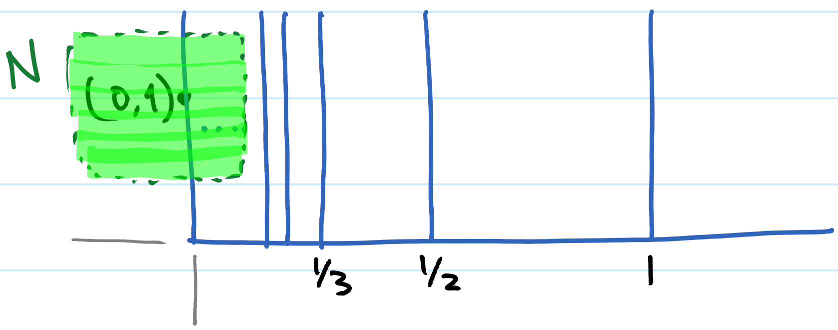
(i) every $U \in \mathcal{U}_x$ is connected

(ii) if $x \in \mathring{N}$ then $\exists\ U \in \mathcal{U}_x$, $x \in U \subset N$

Qu. Give an example of locally connected but disconnected space.

Example. Connected $\not\Longrightarrow$ Locally connected

Let $X = \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0,y) \in \mathbb{R}^2 : y \geq 0\}$
$\cup \{(\frac{1}{n}, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } 0 \leq n \in \mathbb{Z}\}$



X is path connected, $\therefore$ connected

$(0,1) \in X$ has a nbhd $N \cap X$, which does not contain a connected nbhd of $(0,1)$